Domino tilings and determinants

V Aksenov, K. Kokhas

Version 1.3

Contents

1 Introduction

Papers of Zeilberger ([5]), Chaiken and others contain a special technique for solving problems on linear algebra, which allows us to interpret these problems in terms of the graph theory. We are trying to extend this technique and prove the following theorem.

Let F be a simply connected bounded figure on the square grid consisting of unit squares, G_F be its dual graph, i. e. a graph in which vertices correspond to the cells of figure and an edge joins two vertices, iff the corresponding cells share a common side. Denote by A_F the adjacency matrix of graph G_F. We say that two tilings form a good pair, if the difference of numbers of vertical dominoes in these tilings equals 2.

Theorem 1.1. *If the set of all tilings of figure F can be split onto good pairs, then* $\det A_F = 0$*. If the set of all tilings, except one, can be split onto good pairs, then* $\det A_F = (-1)^s$*, where s is one half of the area of the figure.*

In § 2 we describe, how to interpret the determinant in terms of 1-factors and the pfaffian. In § 3 we introduce the notion ¡¡sign of a figure¿¿ and show, how to calculate it. In § 4 we prove the main theorem. In § 5 we calculate the determinant of the adjacency matrices of figures, similar to rectangle.

2 Determinants and 1-factors

Let F be a simply connected figure on the square grid consisting of $2s(F)$ unit squares. Let G_F be the dual graph of the figure F, i. e. vertices of G_F correspond to the cells of figure F and edges correspond to pairs of cells that share common side. Observe that graph G_F is biparite, its partition is determined by chess coloring of figure F. Denote by $A_F = (a_{ij})$ the adjacency matrix of the graph G_F, matrix A_F is symmetrical. We consider also symmetrical matrices of more general type $\widetilde{A}_F$, which can be obtained from A_F if we replace 1's by arbitrary real numbers. For simplicity we interpret graph G_F and its subgraphs as directed graphs, each undirected edge we treat as a pair of edges with opposite directions. Matrix elements a_{ij} we interpret as weights of the corresponding edges.

Let us remind that *1-factor* of directed graph G_F is a subgraph, which has the same set of vertices as graph G_F and such that each its vertex has one ingoing and one outgoing edge.

Let figure F consists of n cells, then $\widetilde{A}_F$ is a matrix of size $n \times n$ and

$$\det \widetilde{A}_F = \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}, \tag{1}$$

where the sum is taken over the set of all permutations of $\{1, \ldots, n\}$, and $\operatorname{sgn}(\pi)$ denotes the sign of a permutation. Each *non-zero* summand in the formula (1) uniquely determines 1-factor of graph G_F, precisely the directed subgraph consisting of edges $i \to \pi(i)$ (if $a_{i,\pi(i)} \neq 0$, then such edge exists). We denote 1-factor, determined by the permutation π, by the same letter π.

If the permutation π is written as a product of cycles, we can calculate its sign by the formula

$$\operatorname{sgn}(\pi) = (-1)^{l_1-1}(-1)^{l_2-1} \ldots (-1)^{l_m-1},$$

where m is the number of cycles in the permutation and l_i is the length of the i-th cycle. Our graph is bipartite, therefore every cycle has even length. Each cycle of the even length considered as a permutation is an odd permutation. Thus, we can calculate the sign of permutation π by the formula

$$\operatorname{sgn}(\pi) = (-1)^{\#\text{the number of cycles in } \pi}.$$

and the determinant can be calculated by the formula

$$\det \widetilde{A}_F = \sum_{\pi} (-1)^{\#\text{the number of cycles in } \pi} W_\pi, \tag{2}$$

where the sum is taken over the set of all 1-factors of graph G_F, and W_π denotes the weight of 1-factor π, which equals by definition to the product of weight of all edges of 1-factor. For adjacency matrix A_F each factor W_π equals 1.

Definition. We will use the term *configuration* instead of *1-factor*, a parity of the number of cycles in the configuration we call a *parity* of configuration, and the expression $(-1)^{\#\text{number of cycles in } \pi}$ we call the *sign* of configuration π.

A domino tilings of the figure we call a *tiling* for brevity. Each tiling of figure F consists of $s(F)$ domino. Each tiling of figure F determines a perfect matching of graph G_F, we will omit the word "perfect" hereinafter.

Definition. Fix the figure F. Denote by c_k the number of tilings of the figure F, which contain exactly k vertical dominoes. We call the polynomial $f_F(x) = \sum_{i=0}^{+\infty} c_k \cdot x^k$ *the polynomial of vertical statistics of tilings of figure* F.

We say that an edge in the configuration is *rising*, if it is vertical and is directed upwards, and *falling*, if it is vertical and is directed downwards. Denote by u_k the number of configurations in figure F, which contain exactly k rising edges. We call the polynomial

$$g_F(x) = \sum_{k=0}^{+\infty} u_k \cdot x^k \tag{3}$$

the polynomial of vertical statistics of configurations of figure F.

Theorem 2.1. *Let F be an arbitrary figure on the square grid consisting of unit squares. Then the following statements hold:*

1) the number of configurations in the graph G_F is equal to the square of the number of tilings of figure F;

2) $g_F(x^2) = f_F^2(x)$.

Proof. 1) Consider a chess coloring of the figure, and split the edges of each configuration onto two groups: edges, which starts from black vertices and edges, which starts from white vertices. Edges of each group determine a matching, which can be interpreted as a tiling. This map is bijective.

2) Due to this bijection we see that the coefficient of x^k in $f_F(x)^2$ is equal to the number of configurations, containing exactly k vertical edges. Since the number of rising edges in each configuration equals one half of the number of vertical edges, the statement follows. □

It follows from the first claim of the previous theorem and the formula (2), that the parity of determinant $\det A_F$ is equal to the parity of the number of tilings of figure F.

Let us remind the definition of pfaffian. For every pair of vertices of undirected graph G we fix the order in which the vertices of this pair should be written. We may assume, that vertices of the graph are numbered and therefore the corresponding order is given for every pair of numbers. The order of pairs vertices allows us to split matchings of the graph onto two classes. Two matchings belong to the same class, if the first matching as a set of ordered pairs of vertices can be transformed to another one by an even permutations. We can mark the matchings of the first class by the plus sign, and mathings from another class by the minus sign. Consider a skew-symmetric matrix A, in which $a_{ij} = -a_{ji}$, if (i,j) — right ordered pair of vertices, which are connected by an edge, and $a_{ij} = 0$ otherwise. We say that edge (i,j) corresponds to matrix element a_{ij}, if (i,j) is a right ordered pair. Define the pfaffian of matrix A as an expression

$$\operatorname{Pf} A = \sum_{\tau} \operatorname{sgn}(\tau) w(\tau),$$

where the summation is taken over the set of all perfect matchings of graph G, and $w(\tau)$ is equal to the product of the matrix elements, corresponding to edges of the matching. It is known that

$$\det A = (\operatorname{Pf} A)^2. \tag{4}$$

For each figure on the square grid and its graph G_F there exists the orientation of the pairs of neighbouring vertices, such that all matchings of figure F have equal sign; this orientation is called *Pfaff* orientation of graph. If A is the skew-symmetrical adjacency matrix of graph G_F, which is given by a Pfaff orientation, then $\operatorname{Pf} A$ is equal to the number of matchings of the graph. But we are interested in another, *non-Pfaff* orientation on the pairs of vertices.

Definition. Consider a chess coloring of vertices of graph G_F. Consider an orientation on the set of pairs of the neighbouring vertices, such that the first vertex in each pair is black. We call this orientation *chess orientation.* Denote by $\widetilde{A}_F^{\#}$ the skew-symmetrical matrix, which is given by this orientation.

Looking at the chess coloring of the figure we can represent $\widetilde{A}_F$ as a block matrix 2×2, left-top block corresponds to black cells, and bottom-right block corresponds to the white cells. If we change signs of all matrix elements, which are written in "white" rows and in "black" columns, we will obtain a skew-symmetrical matrix $\widetilde{A}_F^{\#}$. Thus,

$$\det \widetilde{A}_F = (-1)^{s(F)} \det \widetilde{A}_F^{\#}.$$

It is more convenient for us to explain this formula via pfaffians.

Theorem 2.2. *If F is an arbitrary figure on the square grid with even area $2s(F)$ then*

$$(-1)^{s(F)} \det \widetilde{A}_F = \big(\operatorname{Pf} \widetilde{A}_F^{\#}\big)^2.$$

Proof. The expression $(\operatorname{Pf} \widetilde{A}_F^{\#})^2$ counts all the pairs of matchings (τ_1, τ_2), which are taken with sign $\operatorname{sgn}(\tau_1)\operatorname{sgn}(\tau_2)$. The product of signs is equal to 1, if one matching can be transformed to another by an even permutation, and -1, if the permutation is odd. For the calculation of $\operatorname{sgn}(\tau_1)\operatorname{sgn}(\tau_2)$ let us depict our graph and draw both matchings in it. This picture is a set of cycles, i. e. a configuration, denote it by $\tilde{\pi}$ (the orientation of edges is the same as in the proof of the first point of the theorem 2.1). The map $(\tau_1, \tau_2) \mapsto \pi$ is a bijection. Define the orientation of edges by the checkerboard coloring rule, i.e. the first vertex of an edge is always black. Now construct a permutation which transforms τ_1 in τ_2. First, perform counter-clockwise shift in each cycle, the obtained permutation has parity $(-1)^{\#\text{number of cycles in } \pi}$. As the result of this shift τ_1 becomes τ_2, but all (!) the edges of matching τ_2 are written in the wrong order due to the property of the chess orientation. We will fix it applying transpositions, the parity of the repairing permutation is equal to $(-1)^{s(F)}$. As a result we have

$$\operatorname{sgn}(\tau_1)\operatorname{sgn}(\tau_2) = (-1)^{s(F)+\text{number of cycles in } \pi}. \tag{5}$$

So,

$$\big(\operatorname{Pf} \widetilde{A}_F^{\#}\big)^2 = (-1)^{s(F)} \sum_{\pi} (-1)^{\#\text{number of cycles in } \pi} W_\pi.$$

The sum in the right side is equal to $\det \widetilde{A}_F$ due to (2). □

3 A sign of a simply connected figure on the square grid

Lemma 3.1. *Let P be a simply connected polygon on the square grid. Let a be a number of integer points with an even ordinate and b a number of integer points with an odd ordinate on the boundary of P. Let d be a number of integer points inside P. Then the sum of lengths of vertical sides of the polygon P is equal to $a - b + 2d + 2$ modulo 4.*

Proof. Induction by the area. If the dual graph contains terminal vertex, then cut the corresponding cell. Otherwise we cut a suitable corner cell. □

Remark. We allow some polygons to be degenerate (i. e. cycle on two vertices, with two parallel edges). It is easy to see, that the statement of lemma remains true in degenerate case.

Theorem 3.2. *Let F be a simply connected polygon on the square grid, consisting of even number of cells. Then either for each configuration in graph G_F the parity of the number of rising edges is equal to the parity of the number of cycles in it, or for each configuration these parities are opposite.*

Proof. Consider an arbitrary configuration in G_F. It is obvious, that the number of rising edges in it equals to the number of falling edges, denote this number by v. Let the configuration consists of k cycles. Each cycle is a polygon. Since all of the cycles have even length, and the figure is simply connected, the inner part of each cycle contains even number of integer points. Therefore applying the statement of lemma 3.1 to each cycle, we can omit the term $2d$ in the left hand side of the congruence. Now if we sum up over the set of all cycles, we obtain

$$A - B + 2 \cdot k \underset{\text{mod } 4}{\equiv} \text{the total length of all vertical sides } = 2 \cdot v,$$

where A is equal to the number of integer points with even ordinates and B is the number of integer points with odd ordinates on the boundary of cycles. Since the configuration covers all integer points of the figure, the difference $A - B$ is even and does not depend on configuration. Put $A - B = 2 \cdot t$. Then $2 \cdot v \equiv 2 \cdot t + 2 \cdot k \pmod 4$, and so $v \equiv t + k \pmod 2$. Since t does not depend on configuration, the theorem is proven. □

Definition. If the two parities in the statement of the theorem 3.2 are coincide, we say, that *the sign of figure* F equals 1, otherwise the sign of figure F equal to -1. Thus by definition for each configuration π in graph G_F

$$(-1)^{\text{number of rising edges in } \pi} = \operatorname{sgn} F \cdot (-1)^{\text{number of cycles in } \pi}. \tag{6}$$

We consider also a “logarithm” of the sign of F, which we denote by $\operatorname{Sign} F$. By definition $\operatorname{Sign} F$ is equal to 0 or 1, such that

$$\operatorname{sgn} F = (-1)^{\operatorname{Sign} F}.$$

Lemma 3.3. *The sign of a simply connected figure F can be calculated as follows.*

1) $\operatorname{Sign} F = \frac{1}{2}(A - B)$*, where A is the number of integer points in the dual figure with even ordinates, B is the number of points with odd ordinates.*

2) $\operatorname{Sign} F$ *equals one half of the difference of the number of black vertices and the number of white vertices in the horizontal ¡¡zebra¿¿ coloring of F.*

3) $\operatorname{Sign} F$ *is equal to the parity of the number of horizontal dominoes in any tiling of figure F.*

Proof. 1) It follows from the proof of the theorem 3.2.

2) It is almost the same as the statement 1). The difference $A - B$ equals the difference of numbers of black and white cells in the horizontal ¡¡zebra¿¿ coloring of F.

3) If we interpret the tiling as a configuration, the number of cycles in it equals the number of dominoes, the number of rising edges is equal to the number of vertical dominoes. By definition $\operatorname{sgn} F = -1$, if the parity of the number of the rising edges in the configuration is not equal to the parity of the number of

cycles, and $\operatorname{sgn} F = 1$ otherwise. Thus,

$$\text{Sign}\, F \underset{\text{mod } 2}{\equiv} \text{number of cycles} + \text{number of rising edges} =$$
$$= \text{number of dominoes} + \text{number of vertical dominoes.}$$

It remains to observe, that vertical dominoes are counted in both summands, while horizontal dominoes is counted only in the first one. □

In the definition of pfaffian we can arbitrarily set the sign "plus" to the one of the two classes of matchings, and the sign "minus" to the other one. In the case of chess orientation we can set this signs "geometrically". Denote by $V(\tau)$, $H(\tau)$ the number of vertical and horizontal edges in matching τ.

Lemma 3.4. *In the definition of pfaffian we can state that the sign of matching τ equals $(-1)^{\frac{1}{2}(H(\tau)+\text{Sign}\, F)}$.*

Proof. The sum $H(\tau)+\text{Sign}\, F$ is even by lemma 3.3. Consider two matchings τ_1, τ_2 and configuration π, which is determined by them as in the proof of theorem 2.2. Let us remind, that each matching contains $s(F)$ edges, numbers $V(\tau_1)$ and $V(\tau_2)$ always have the same parity, and the number of rising edges in the configuration, given by the two matchings, equals $\frac{1}{2}(V(\tau_1)+V(\tau_2))$.

Let's check that signs, specified by the statement of the lemma, are in concordance with the parity of the permutation from the definition of a sign of a matching, i. e. the sum $s(F) +$ number of cycles π in equality (5) is even if and only if the signs are equal. It is true, because modulo 2 we have

$$\begin{aligned}\text{number of cycles in } \pi + s(F) &= \text{number of rising edges in } \pi + \text{Sign}\, F + s(F) = \\ &= \frac{1}{2}(V(\tau_1)+V(\tau_2)) + \text{Sign}\, F + \frac{1}{2}(V(\tau_1)+H(\tau_1)+V(\tau_2)+H(\tau_2)) \equiv \\ &\equiv \frac{1}{2}(H(\tau_1)+\text{Sign}\, F) + \frac{1}{2}(H(\tau_2)+\text{Sign}\, F).\end{aligned}$$

□

4 Formulae for the determinant of the adjacency matrix

The following theorem reduces the question of calculation of $\det A_F$ to the investigation of the vertical statistics of figure F.

Theorem 4.1. *For every simply connected figure F*

$$\det A_F = \operatorname{sgn} F \cdot \sum_{\pi} (-1)^{\textit{the number of rising edges in } \pi}, \tag{7}$$

$$\det A_F = \operatorname{sgn} F \cdot g_F(-1) = \operatorname{sgn} F \cdot f_F^2(\mathbf{i}), \tag{8}$$

where g_F and f_F are polynomials of the vertical statistics.

Proof. Comparing formulae (2) and (6), we obtain automatically (7). By formulae (7) and (3)

$$\det A_F = \operatorname{sgn} F \cdot \sum_{\pi} (-1)^{\text{the number of rising edges in } \pi} = \sum_{i=0}^{+\infty} c_i \cdot (-1)^i = g_F(-1).$$

Substituting -1 instead of x^2 in g, we obtain $\det A_F = \operatorname{sgn} F \cdot g_F(-1) = \operatorname{sgn} F \cdot {f_F}^2(\mathbf{i})$. □

Definition. We say that a pair of tilings is *good*, if the difference of numbers of vertical dominoes in them is equal to 2.

Theorem 4.2. *Let F be an arbitrary simply connected figure on the square grid consisting of $2s(F)$ squares. If the set of all tilings of figure F can be split onto good pairs, then* $\det A_F = 0$*. If the set of all tilings, except one, can be split onto good pairs, then* $\det A_F = (-1)^{s(F)}$*.*

Proof. Let us calculate $\det A_F$ by the formula (8). If the good pair consists of the tiling with k vertical dominoes and the tiling with $k+2$ vertical dominoes, then its contribution into $f_F(\mathbf{i})$ is equal to $\mathbf{i}^k + \mathbf{i}^{k+2} = 0$. Therefore all good pairs contribute zero into $f_F(\mathbf{i})$ and the first claim of the theorem follows.

If the set of all tilings, except one, can be split onto good pairs, we denote the number of vertical and horizontal dominoes in the remaining tiling by v and h, $h + v = s(V)$. Then $f_F(\mathbf{i}) = \mathbf{i}^v$ by the previous reasoning, $\operatorname{sgn} F = (-1)^h$ by the lemma 3.3, and therefore $\det A_F = \operatorname{sgn} F \cdot {f_F}^2(\mathbf{i}) = (-1)^{h+v} = (-1)^{s(F)}$. □

So for the calculation $\det A_F$ we should know whether the set of all tilings of the figure can be split onto good pairs. The figure (not simply connected) for which the set of tilings can not be split onto good pairs is depicted on figure 1.

It is clear from the proof that in terms of the vertical statistics the possibility to split the set of tilings onto good pairs is equivalent to the divisibility of the polynomial $f_F(x)$ by $x^2 + 1$.

Before we move to an application of the theorem, we shall prove, that for an arbitrary simply connected figure F on the square grid of even number of squares $\det A_F = 0$ or $\det A_F = \pm 1$. This statement will follow from the next lemma.

Lemma 4.3. *The set of all tilings of an arbitrary simply connected figure on the square grid of even number of squares could be split onto good pairs, probably except one.*

Proof. To prove this theorem we use the method of mathematical induction on the area of the figure F.

Base. F contains 2 squares, i.e. F is the domino. It is obvious, that the theorem holds for F.

Step.

To start we choose the "lowest" "corner" square of F, from which we shall apply halfdiagonal lemma. For that we look at all bottom-right diagonals from all squares of F, then choose the corner one of two types: $\ulcorner$ or $\lrcorner$, which lies on the lowest such diagonal. Without loss of generality, let it be $\ulcorner$. Let us apply halfdiagonal lemma from the chosen squares. There could be three cases:

The first case. The squares, which are neighbours by side to the right and to the bottom to the last

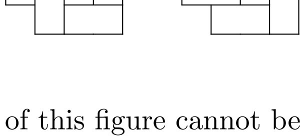
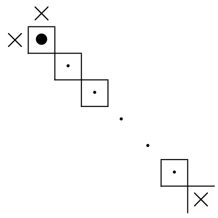


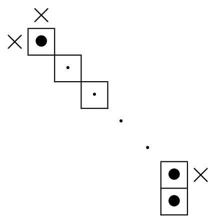
Figure 1. The tilings of this figure cannot be split onto good pairs

square of halfdiagonal, belong to F.



Note, that this is not possible to happen, because there are obviously exist such corner $\lrcorner$, which lies on lower diagonal, than the chosen corner.

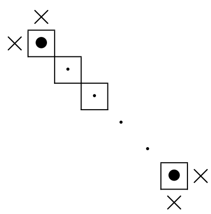
The second case. F contains only one of two neighbours by side to the right and to the bottom to the last square of halfdiagonal.



Applying halfdiagonal lemma, the domino marked by black circles could be removed, because the set of all tilings, which doesn't contain it, could be split onto good pairs. To complete this case we use the induction hypothesis.

It is possible, that after domino removal, F was split onto different parts. The set of all tilings in F is the cartesian product of the set of all tilings of created parts. If at least in one part the set of all tilings could be split onto good pairs, then the set of all tilings of F could be split too. Otherwise, the cartesian product of unpaired tiling in all parts is the only unpaired tiling of F.

The third case. The squares, neighbours by side to the right and to the bottom of the last square of halfdiagonal, aren't in F.



By the reasoning, similar to the proof of the lemma of halfdiagonal, it is simple, that in this case the set of all tilings of F could be split onto good pairs. $\square$

As the corollary of this lemma and Theorem 4.2 we get:

Theorem 4.4. *For an arbitrary simply connected figure F on the square grid of even number of squares* $\det A_F = 0$ *or* $\det A_F = \pm 1$.

5 *Application to "stamps" and rectangles*

Definition. We call *n-stamp* a figure that can be obtained from square $n \times n$, by deleting some cells on its upper and its right sides (so it looks like postage stamp but with irregular perforation along its two sides). Let us enumerate rows of a n-stamp from bottom to top, and columns from left to right. Each cell is determined by the numbers of its row and column. We say that n-stamp is *regular* (figure 4), if it contains exactly one cell from the pair (n, i) and (i, n), if $i < n$, and does not contain the cell (n, n). Otherwise we say that the stamp is *irregular* (See figure 2).

Stamps were introduced by D.Karpov [2], they are interesting, because we know the parity of their number of tilings, precisely, the following theorem holds.

Theorem 5.1 ([2]). *The number of tilings of n-stamp is odd if and only if the stamp is regular.*

Lemma 5.2 (about a ¡¡halfdiagonal¿¿). *Let figure F contains three diagonal rows of cells, like on the figure 3, and cells, which are marked by crosses, do not belong to the figure. Then the set of all tilings of figure F, which does not contains domino marked with bold circles, can be split onto good pairs.*

We say that the lemma statement holds for the bottom-right direction. We will apply this lemma in other diagonal directions, too. This lemma in slightly different form, is proven in [3, lemma 2], it has been applied there for the proof of the theorem 5.1. We apply the similar reasoning in the following theorem.

Theorem 5.3. *1) Let F be an arbitrary regular n-stamp. Then* $\det A_F = (-1)^{n(n-1)/2}$.

2) If F is an irregular n-stamp, then $\det A_F = 0$.

Proof. 1) The expression $n(n-1)$ in the formula equals the area of any regular stamp. By theorem 4.2 it is sufficient to check, that the set of all tilings of each regular stamp, except one, can be split onto good pairs. We will check it by the induction by n. The base is trivial.

Step of induction, $n \to n+1$. Consider a regular $(n+1)$-stamp. We will split the set of its tilings onto good pairs. For this we take a look at the bottom-right and upper-left corner cells of the $(n+1)$-stamp. One of these cells lies inside the $n \times n$ square, let it be the upper-left cell. Apply the halfdiagonal lemma in the bottom-right direction starting from this cell. Then the set of tilings, which does not contain the marked domino (figure 4, left) can be split onto good pairs. Let's look at tilings, which contain this domino. Apply the halfdiagonal lemma again in the upper-left direction starting from the cell to the left of the marked domino (figure 4, middle). By this lemma the set of tilings, which does not contain the marked domino in the upper-left corner, can be split onto good pairs. If we look at the remaining tilings, they contain this domino. Apply the halfdiagonal lemma once again in the bottom-right direction from the cell below the domino and so on. As a result of numerous applying the halfdiagonal lemma we split the set of tilings onto pairs except the tilings, containing all the dominoes on the left and the bottom sides of our $(n+1)$-stamp (fig. 4, right). By the induction hypothesis there is the bijection between the remaining tilings and tilings of the regular n-stamp. Therefore all tilings except one can be split onto good pairs.

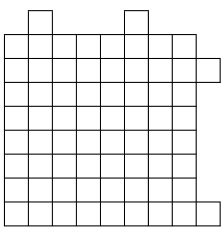


Figure 2. Irregular 9-stamp

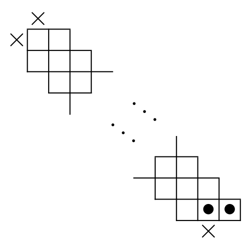


Figure 3. Halfdiagonal

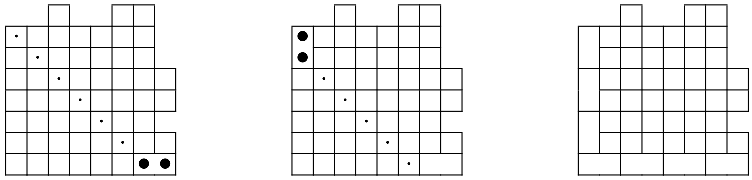


Figure 4. Construction of ¡¡unpaired¿¿ tiling of $(n+1)$-stamp

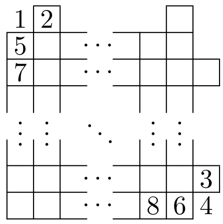


Figure 5. Layout of the cells for the irregular stamp

2) By theorem 4.2 it is sufficient to check, that tilings of each irregular stamp can be split onto good pairs. We will check it by induction by n. The base is trivial.

Step of induction, $n-1 \to n$. Consider an arbitrary n-stamp. We mark some cells of its $n \times n$ square like on the figure 5.

Consider the following cases:

1) The figure does not contain cells 1 and 4. Consider a diagonal from 5 to 6. By the halfdiagonal lemma the set of all tilings can be split onto pairs (because the marked domino does not belong to the figure). Similarly, if four cells 1, 2, 3, 4 don't belong to the stamp.

2) The stamp contains the cell 1, but not the cell 4 (or vice versa). Consider the first case, the second one is similar. Apply the halfdiagonal lemma in the direction from 6 to 1. As in the proof of the previous item, we split the set of all tilings onto pairs, except those tilings for which the position of dominoes on the leftmost column and bottom row is fixed like on the figure 4, right. The set of exceptional tilings can be split onto good pairs by induction hypothesis.

3) Cells 1 and 4 belong to the stamp, but 2 and 3 do not belong. Then each tiling contains dominoes $1-5$ and $4-6$. Cut them. By the halfdiagonal lemma, which we apply in the direction from 7 to 8, the set of all tilings can be split onto good pairs.

4) Cells 1, 2, 4 belong to the stamp, but 3 does not belong (or similarly 1, 3, 4 belong to the stamp, but 2 does not belong). Obviously, each tiling contains the domino $4-6$. Cut it. Apply the halfdiagonal lemma in the direction from 8 to 7. Observe that each tiling contains the domino $5-7$ and therefore each tiling contains the domino $1-2$. We cut these dominoes and finish the proof by induction, like in item 2. □

Remark that this proof proves also the theorem 5.1. The following criteria of parity of tilings of the rectangle is proven in [2].

Theorem 5.4. *The number of tilings of the rectangle $n \times m$ is odd if and only if numbers $n+1$ and $m+1$ are coprime.*

In [3] this theorem is proven exactly by spliting tilings onto good pairs! Combining this theorem and the theorem 4.2, we obtain the following theorem.

Consequence 5.4.1. *For an arbitrary rectangle* $m \times n$

$$\det A_{n\times m} = \begin{cases} 0, & \text{if } (n+1, m+1) \neq 1; \\ (-1)^{\frac{n\cdot m}{2}}, & \text{if } (n+1, m+1) = 1; \end{cases}$$

where (n, m) *is the greatest common divisor of* n *and* m.

This result is already known, see for example [6]. But the combinatorial proof, presented here, is new.

References

[1] *Vyalyj M. N.* Pfaffians or an art to place signs... // Math. Prosveschenie. Ser. 3rd. 2005. No. 9. P. 129–142.

[2] Karpov D. V. On parity of number of domino tilings. Not published. 1997.

[3] *Kokhas K. P.* Domino tilings // Math. Prosveschenie. Ser. 3rd. 2005. No. 9. P. 143–163.

[4] *Kokhas K. P.* Domino tilings of aztec diamonds and squares // Zapiski nauchn. semin. POMI. 2008. Vol. 360. P. 180–230.

[5] *Zeilberger D.* A combinatorial approach to matrix algebra // Discrete Mathematics 56 (1985). . 61–72. North-Holland.

[6] *Pragel D.* Determinants of box products of paths // `http://arxiv.org/1110.3497`