Theorem 1 (Chebyshev's Inequality). *If $f, g : [a,b] \to \mathbb{R}$ are two monotonic functions of the same monotonicity, then*

$$(b-a) \cdot \int_a^b f(x)g(x)\,dx \geq \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right).$$

If f and g are of opposite monotonicity, then the inequality should be reversed.

Theorem 2 (The Mean Value Theorem). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function and $g : [a,b] \to \mathbb{R}$ be a non-negative integrable function. Then, there is $c \in [a,b]$ such that*

$$f(c) \cdot \int_a^b g(x)dx = \int_a^b f(x)g(x)\,dx.$$

Theorem 3 (Cauchy-Schwarz's Inequality). *If $f, g : [a,b] \to \mathbb{R}$ are integrable, then*

$$\left|\int_a^b f(x)g(x)\,dx\right|^2 \leq \left(\int_a^b f^2(x)\,dx\right) \cdot \left(\int_a^b g^2(x)\,dx\right),$$

with equality when $|g(x)| = c \cdot |f(x)|$

Theorem 4 (Hölder's Inequality). *If $f, g : [a,b] \to \mathbb{R}$ are integrable and $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$, then*

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} \cdot \left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}},$$

with equality when $|g(x)| = c \cdot |f(x)|^{p-1}$.

1. a) Prove Chebyshev's inequality.
b) Let $f : [0,1] \to \mathbb{R}$ be a non-decreasing continuous function and n be a positive integer. Prove that $\int_0^1 f(x)\mathrm{d}x \leq (n+1) \cdot \int_0^1 x^n f(x)\mathrm{d}x$.

c) Let $f : [0,1] \to (0,1)$ be a Riemann integrable function. Show that $\frac{2\int_0^1 xf^2(x)\mathrm{d}x}{\int_0^1 (f^2(x)+1)\mathrm{d}x} < \frac{\int_0^1 f^2(x)\mathrm{d}x}{\int_0^1 f(x)\mathrm{d}x}$.

2. Let $f \in C^2[0,1]$. Show that, for any $y \in [0,1]$, $|f'(y)| \leq 4\int_0^1 |f(x)|\mathrm{d}x + \int_0^1 |f''(x)|\mathrm{d}x$.

3. Let $f : [1,13] \to \mathbb{R}$ be a convex and integrable function. Prove that $\int_1^3 f(x)\mathrm{d}x + \int_{11}^{13} f(x)\mathrm{d}x \geq \int_5^9 f(x)\mathrm{d}x$.

4. Let $f : [0,1] \to [0,\infty)$ be integrable. Prove that $2\int_0^1 f^4(x)\mathrm{d}x + \left(\int_0^1 f(x)\mathrm{d}x\right)^4 \geq 3\left(\int_0^1 f^2(x)\mathrm{d}x\right)^2$.

5. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $0 \leq f'(x) \leq 1$ and $f(a) = 0$. Prove that $3\left(\int_a^b f^2(x)\mathrm{d}x\right)^3 \geq \int_a^b f^8(x)\mathrm{d}x$.

6. Let $f : [1,\infty) \to \mathbb{R}$ and $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$. Prove that there exist two positive numbers c_1 and c_2, such that $\frac{c_1}{x} \leq f(x) \leq \frac{c_2}{x}$ for $x \in [1,\infty)$.

7*. Let f be a differentiable function on $[1,2]$ with $f(1) = 1$, $f(2) = 2$ and $f'(x) + f(x) > 1$ for every $x \in [1,2]$. Prove that $1 \leq \int_1^2 f(x)\mathrm{d}x \leq e$.

8*. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative and $\int\limits_0^1 f(x)\mathrm{d}x = \int\limits_0^1 xf(x)\mathrm{d}x = 1$. Prove that $\int\limits_0^1 |f'(x)|^3\mathrm{d}x \geq \left(\frac{128}{3\pi}\right)^2$.

9. Let $f(x) \in C^2[0,1]$, $|f''(x)| \leq 1$ and $f(x)$ reach its maximum value $\frac{1}{4}$ on $(0,1)$. Prove that $|f(0)| + |f(1)| \leq 1$.

10. Let $f(x)$ be continuous in $[0,1]$, $\int\limits_0^1 f(x)\mathrm{d}x = 0$ and $\int\limits_0^1 xf(x)\mathrm{d}x = 1$. Prove that there exists at least one point c such that $|f(c)| > 4$.

11*. Let $f : [a,b] \to \mathbb{R}$ be twice differentiable with continuous derivative f'' and $f(a) = f(b)$. Prove that $\left(\int\limits_a^b xf'(x)\mathrm{d}x\right)^2 \leq \frac{(b-a)^5}{120} \int\limits_a^b (f''(x))^2\mathrm{d}x$.