Theorem (Mean Value Theorem). *If f is a continuous function on a closed interval $[a,b]$, and differentiable on the open interval (a,b), then there exists a point $c \in (a,b)$ such that:*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

1. Find all continuous functions that satisfy $f(xy) = f(x)f(y) - f(x+y) + 1$ and $f(1) = 2$.
2. For $f : \mathbb{R} \to \mathbb{R}$ and any $a < b$ $|f(a) - f(b)| < |a - b|$. Prove that if $f(f(f(0))) = 0$ then $f(0) = 0$.
3. Find all $f(x)$ such that $f(x) : \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$ and $f(x) + f(\frac{x-1}{x}) = 1 + x$.
4. Given $f : \mathbb{Z}^+ \to \mathbb{Z}^+ \cup \{0\}$ with $f(2) = 0$, $f(3) > 0$, $f(9999) = 3333$ and for any n and m $f(m+n) = f(m) + f(n) + \delta_{m,n}$ where $\delta_{m,n} \in \{0,1\}$, find $f(2013)$.
5. For continuous function $f : \mathbb{R} \to \mathbb{R}$ $f(x)$ is rational for any irrational x. Find all such $f(x)$.
6. Find all continuously differential functions $f(x)$ such that $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for any x.
7. Let f be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.
8. Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0,a]$ with the property that the region $R = \{(x,y) \,|\, 0 \leq x \leq a, 0 \leq y \leq f(x)\}$ has perimeter k units and area k square for some real k.
9. Find all at least twice differentiable functions $f : (0,+\infty) \to (0,+\infty)$ for which there is a positive real number a such that $f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$ for all $x > 0$.
10. Let $a > 0$ and $f : [-a,a] \to \mathbb{R}$ be twice differentiable function with the property $|f(x)| \leq 1$ for all $x \in [-a,a]$. Prove that if for $p,q \geq 2$ $(f(0))^p + (f'(0))^q > 1 + \left(\frac{2}{a}\right)^q$ then there exist $c \in (-a,a)$ such that $p(f(c))^{p-1} + q(f'(c))^{q-2}f''(c) = 0$.
11. Let $f : [0,2] \to \mathbb{R}$ be a differentiable function having a continuous derivative and satisfying $f(0) = f(2) = 1$ and $|f'| \leq 1$. Show that $\left|\int_0^2 f(x)\,\mathrm{d}x\right| > 1$.

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