1. $f(xy) = f(x)f(y) - f(x+y) + 1$ and $f(1) = 2$.

Put $x = 1$ and $y = n$: $f(n) = 2f(n) - f(n+1) + 1 \Rightarrow f(n+1) = f(n) + 1$. This means that $f(n) = n + 1$ (1).

For $x = k$ and $y = \frac{n}{k}$: $f(n) = f(k)f(\frac{n}{k}) - f(k + \frac{n}{k}) + 1 \Rightarrow n + 1 = (k+1)f(\frac{n}{k}) - k - f(\frac{n}{k}) + 1 \Rightarrow f(\frac{n}{k}) = \frac{n}{k} + 1$.

Thus, for $x \in \mathbb{Q}$ $f(x) = x + 1$. And since $\mathbb{Q}$ is dense in $\mathbb{R}$ we have $f(x) = x + 1$ for $x \in \mathbb{R}$.

2. Let $f(0) \neq 0$. Then $|f(0) - 0| > |f(f(0)) - f(0)| \geq |f(f(f(0))) - f(f(0))| \geq |0 - f(f(0))| \geq |f(0) - f(f(f(0)))| = |f(0) - 0|$. Contradiction.

3. $f(x) + f(\frac{x-1}{x}) = 1 + x$ (1).

For $x = \frac{x-1}{x}$: $f(\frac{x-1}{x}) + f(\frac{-1}{x-1}) = \frac{2x-1}{x}$ (2).

For $x = \frac{-1}{x-1}$: $f(\frac{-1}{x-1}) + f(x) = \frac{x-2}{x-1}$ (3).

(1) + (3) − (2): $2f(x) = 1 + x + \frac{x-2}{x-1} - \frac{2x-1}{x} = \frac{x^3-x^2-1}{x(x-1)} \Rightarrow F(x) = \frac{x^3-x^2-1}{2x(x-1)}$.

4. $3333 = f(9999) = f(9996) + f(3) + \delta_{9996,3} = f(9993) + 2f(3) + \delta_{9993,3} = \ldots = 3333f(3) + \delta_{9996,3} + \ldots$

This means, that $f(3) = 1$ and $\delta_{9996,3} = \delta_{9993,3} = \ldots = \delta_{3,3} = 0$. Since 2013 is divisible by 3, $f(2013) = 671$.

5. Since the function is continuous it maps $(-\infty, +\infty)$ to some interval X. All irrationals from $(-\infty, +\infty)$ are mapped into rationals from X. Thus, rationals from $(-\infty, +\infty)$ are mapped to all irrationals and some set of rationals from X. This is impossible, since the set of irrationals in X is incountable, while the set of rationals is countable.

6. Let f be not equal to zero. Then, since f is continuous, then there exists an interval $[a, b]$ such that $|a - b| < 1$, $f(a) = 0$ and $|f(b)|$ is the maximum on the interval. By the Mean Value Theorem: $\frac{f(b)-f(a)}{b-a} = f'(c) \Rightarrow |f(c)| < |f(b)| < |f'(c)|$. Contradiction.

7. Let $g(t) = \frac{f(\cos t)}{\sin t}$, then $g(t + \pi) = g(t)$.

$g(2t) = \frac{f(2\cos^2 t - 1)}{2\sin t \cos t} = \frac{f(\cos t)}{\sin t} = g(t)$.

Then for any n and k $g(1 + \frac{n\pi}{2^k}) = g(2^k + n\pi) = g(2^k) = g(1)$. Since, $\{1 + \frac{n\pi}{2^k} \mid n, k \in \mathbb{Z}\}$ is dense and g is continuous on its domain, g is constant on its domain. We know that $g(t) = g(-t)$, thus $g(t) = 0$, when t is not a multiple of π. Hence, $f(x) = 0$ for $x \in (-1, 1)$. Finally, since f is continuous, $f(x) = 0$ for $x \in [-1, 1]$.

8. When $a > 2$, $f(x) = \frac{2a}{a-2}$ satisfies: the perimeter and the area are equal to $\frac{2a^2}{a-2}$.

Now, suppose $a \leq 2$. Let M be the maximal value of $f(x)$. Then the area does not exceed $a \cdot M$. At the same time, the perimeter is at least $2M + a$: from $(0, 0)$ to point with $f(x) = M$, from point with $f(x) = M$ to $(a, 0)$ and from $(a, 0)$ to $(0, 0)$. It can be seen that area $\leq a \cdot M \leq 2M < 2M + a \leq$ perimeter.

9. $f'(x) = \frac{a}{xf(\frac{a}{x})}$. Let us take the derivative: $f''(x) = -\frac{a}{x^2 f(\frac{a}{x})} + \frac{a^2 f'(\frac{a}{x})}{x^3 f^2(\frac{a}{x})}$.

Now, substitute $f(\frac{a}{x}) = \frac{xf'(x)}{a}$ and $f'(\frac{a}{x}) = \frac{x}{f(x)}$: $f''(x) = -\frac{f'(x)}{x} + \frac{f'(x)^2}{f(x)}$.

Clear denominators: $xf(x)f''(x) + f(x)f'(x) = xf'(x)^2$.

Divide by $f(x)^2$: $0 = \frac{f'(x)}{f(x)} + \frac{xf''(x)}{f(x)} - \frac{xf'(x)^2}{f(x)^2} = \left(\frac{xf'(x)}{f(x)}\right)'$.

Thus, $\frac{f'(x)}{f(x)} = \frac{d}{x}$ and $f(x) = cx^d$.

10. By the Mean Value Theorem there exists $c_1 \in [-a, 0]$ such that $|f'(c_1)| = \frac{|f(0)-f(-a)|}{0-(-a)} \leq \frac{2}{a}$ and, consequently, $f(c_1)^p + f'(c_1)^q \leq 1 + \left(\frac{2}{a}\right)^q$. Analogously, there exists $c_2 \in [0, a]$ such that $f(c_2)^p + f'(c_2)^q \leq 1 + \left(\frac{2}{a}\right)^q$. Thus, there exists $c \in [c_1, c_2]$ such that $(f(c)^p + f'(c)^q)' = 0$. This is almost what we need, except for a multiplicative factor $f'(c)$. We can divide by it only if $f'(c) = 0$. However, if $f'(c) = 0$ then $f(c)^p + f'(c)^q = f(c)^p \leq 1 < f(0)$, but it should be the maximum on $[c_1, c_2]$.

11. By the Mean Value Theorem there exists $c \in [0, x]$ such that $-1 \leq f'(c) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)-1}{x}$ and $f(x) \geq 1 - x$. Analogously, $1 \geq f'(c) = \frac{f(2)-f(x)}{2-x} = \frac{1-f(x)}{2-x}$ and $f(x) \geq x - 1$. Thus, $f(x) \geq |x - 1|$. $\int\limits_0^2 f(x)\,\mathrm{d}x \geq \int\limits_0^2 |x-1|\,\mathrm{d}x = 1$. However, $|x - 1|$ is not continuous, thus, the strong inequality follows.